

Bose Condensates in TOP Traps Exhibit Circulating Superfluid Flows

Juhao Wu and A. Widom ^a

^aPhysics Department, Northeastern University, Boston MA 02115

For spin one atoms localized in a quadrapole magnetic field gradient, the atoms may be impeded from spin flipping their way out from the center of the trap by the application of a rotating uniform magnetic field. From a quantum mechanical viewpoint, such a trap for a Bose condensate is equivalent to having a superfluid in a rotating bucket. Vorticity is then expected to be induced in the condensate fluid flow without the application of any further external perturbations.

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The two fluid model[1] has been quite successful for describing liquid He^4 below the lambda temperature. The model asserts that the normal fluid component can easily carry vorticity, but the superfluid component can undergo only potential flows with velocity $\mathbf{v}_s = \nabla\Phi$. However, it is presently understood that when superfluid He^4 is located in a rotating container, vorticity can enter into the superfluid flow in the form of “vortex lines” [2,3] with a circulation $\kappa = \oint \mathbf{v}_s \cdot d\mathbf{r}$ quantized in units of $\kappa_0 = (2\pi\hbar/M)$.

In the light of recent progress in preparing Bose condensates in magnetic traps[4–6], there has been considerable interest in whether or not vorticity can play an important role in the experimental features of a mesoscopic superfluid. Our purpose is to point out that superfluid vorticity *must already be present* in those experiments which employ rapidly rotating magnetic fields, i.e. in the so-called “TOP trap” experiments. The detailed description of TOP traps will be reviewed in the work which follows. Here, we note that the implications of rotational vorticity for TOP trap experiments have *not* been previously explored.

Consider (at first) the quantum dynamic behavior of (say) superfluid He^4 in an arbitrarily shaped rotating bucket. Let H denote the Hamiltonian of the superfluid in the bucket *if the bucket were not rotating*. The Hamiltonian H does not depend on time. However, since the bucket does rotate at angular velocity

$$\mathbf{\Omega} = \mathbf{n}(d\theta/dt), \quad (1)$$

where \mathbf{n} is a unit vector along the axis of rotation, the fluid laboratory frame Hamiltonian actually develops a time dependence given by

$$H(t) = S(t)HS^\dagger(t), \quad (2)$$

$$S(t) = \exp(-i\mathbf{n} \cdot \mathbf{L}\theta(t)/\hbar), \quad (3)$$

where \mathbf{L} is the total angular momentum of the fluid. All that is required to understand the above time dependent Hamiltonian $H(t)$ for a fluid in a rotating bucket is that the angular momentum of the fluid is the generator of rotations[7].

A crucial step is to *eliminate the time dependence* from the Hamiltonian $H(t)$ by using the following quantum mechanical canonical transformation ($H(t) \rightarrow \mathcal{H}$),

$$\mathcal{H} = S^\dagger(t)H(t)S(t) - i\hbar S^\dagger(t)(\partial S(t)/\partial t), \quad (4)$$

to a frame rotating along with the bucket. Employing Eqs.(1)-(4), one finds

$$\mathcal{H} = H - \mathbf{\Omega} \cdot \mathbf{L}. \quad (5)$$

Note that the effective Hamiltonian \mathcal{H} in Eq.(5) does *not* depend on time if the angular velocity $\mathbf{\Omega}$ does not depend on time. We are thereby perfectly justified[8] in using \mathcal{H} in Eq.(5) to describe a thermal equilibrium Bose fluid within a rotating bucket; The Gibbs canonical distribution is

$$\rho_{eq} = \exp(-(F - H + \mathbf{\Omega} \cdot \mathbf{L})/k_B T) \quad (6)$$

The notion of thermal equilibrium does not appear so simple before the transformation from the

laboratory frame (where $H(t)$ depends on time) to the rotating frame (where \mathcal{H} does not depend on time). It is important to realize that Eq.(5) represents an entirely rigorous Hamiltonian which describes exactly why a rotating bucket at angular velocity Ω induces vorticity and net angular momentum into the fluid contained within the bucket.

The transition from a laboratory frame TOP trap time varying Hamiltonian $H(t)$ to a rotating frame *time independent Hamiltonian* \mathcal{H} is a bit more subtle but will now be shown to be closely analogous.

A TOP trap for triplet (spin one) atoms is simply a time dependent magnetic field $\mathbf{B}(\mathbf{r}, t)$ constructed by superimposing a homogeneous rotating magnetic field $\mathbf{B}_h(t)$ together with a non-rotating inhomogeneous quadrupole magnetic field $\mathbf{B}_Q(\mathbf{r})$; In detail,

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_h(t) + \mathbf{B}_Q(\mathbf{r}), \quad (7)$$

where the quadrupole field is given in terms of the field gradient amplitude G ,

$$\mathbf{B}_Q(\mathbf{r}) = G(\mathbf{r} - 3(\mathbf{n} \cdot \mathbf{r})\mathbf{n}). \quad (8)$$

The homogeneous field \mathbf{B}_h rotates about (and is normal to) the rotation axis unit vector \mathbf{n} ; In detail, $\mathbf{n} \cdot \mathbf{B}_0 = 0$ and

$$\mathbf{B}_h(t) = \mathbf{B}_0 \cos(\Omega t) + \mathbf{n} \times \mathbf{B}_0 \sin(\Omega t). \quad (9)$$

A triplet state (spin one) atom interacts with the magnetic field \mathbf{B} so that there exists three possible (potential) energy levels, namely zero energy and $\pm \hbar \gamma |\mathbf{B}|$, where γ denotes the magnitude of the gyro-magnetic ratio. An atom in only one of these states can be trapped with the potential

$$U(\mathbf{r}, t) = \hbar \gamma |\mathbf{B}(\mathbf{r}, t)|, \quad (10)$$

The trap potential in cylindrical coordinates $\mathbf{r} = (\rho, \varphi, z)$ (where $z = \mathbf{n} \cdot \mathbf{r}$) is determined by

$$|\mathbf{B}| = G \sqrt{\rho^2 + b^2 + 2b\rho \cos(\varphi - \Omega t) + 4z^2}. \quad (11)$$

The magnetic length of the TOP trap is defined as $b = |\mathbf{B}_0|/G$.

For N atoms in a TOP trap, the time dependent Hamiltonian has the form

$$H(t) = \sum_{1 \leq j \leq N} h_j(t) + \sum_{1 \leq j \leq k \leq N} v_{jk}. \quad (12)$$

The two body pair interaction on the right hand side of Eq.(12) is assumed to conserve total angular momentum. but the time dependent single particle Hamiltonian sum on the right hand side of Eq.(12) *does not conserve total angular momentum* $\mathbf{n} \cdot \mathbf{L}$; i.e.

$$h_j(t) = -\left(\frac{\hbar^2}{2M}\right) \nabla_j^2 + U(\rho_j, \varphi_j - \Omega t, z_j) \quad (13)$$

has a potential which depends on the angle φ_j , as well as ρ_j and z_j .

One may now solve the Schrödinger equation for the many body wave function χ ,

$$i\hbar \left(\frac{\partial \chi(\dots, \mathbf{r}_j, \dots, t)}{\partial t} \right) = H(t) \chi(\dots, \mathbf{r}_j, \dots, t), \quad (14)$$

by looking for a solution of the form

$$\chi(\dots, \rho_j, \varphi_j, z_j, \dots, t) = \Psi(\dots, \rho_j, \varphi_j - \Omega t, z_j, \dots, t) \quad (15)$$

yielding

$$i\hbar \left(\frac{\partial \Psi(\dots, \mathbf{r}_j, \dots, t)}{\partial t} \right) = \mathcal{H} \Psi(\dots, \mathbf{r}_j, \dots, t). \quad (16)$$

The new Hamiltonian \mathcal{H} *does not depend on time*,

$$\mathcal{H} = \sum_{1 \leq j \leq N} \tilde{h}_j + \sum_{1 \leq j \leq k \leq N} v_{jk}, \quad (17)$$

$$\tilde{h}_j = -\left(\frac{\hbar^2}{2M}\right) \nabla_j^2 + U(\mathbf{r}_j) + i\hbar \Omega \left(\frac{\partial}{\partial \varphi_j} \right). \quad (18)$$

Eqs.(17) and (18) yield the Hamiltonian form

$$\mathcal{H}(\text{TOP trap}) = H - \Omega \cdot \mathbf{L} = H - \Omega L_z \quad (19)$$

which *does not depend on time*, *does not conserve angular momentum* but does represent the central result of this work. In the rotating frame of the TOP trap, the single atom potential does not depend on time

$$U(\mathbf{r}) = \hbar \gamma G \sqrt{\rho^2 + b^2 + 2b\rho \cos \varphi + 4z^2}, \quad (20)$$

and does not conserve the angular momentum component $L_z = \mathbf{n} \cdot \mathbf{L}$.

In the previous theoretical literature concerning Bose condensates in TOP traps[9,10], a time averaged Hamiltonian was employed

$$\langle H \rangle = \sum_{1 \leq j \leq N} \langle h_j \rangle + \sum_{1 \leq j \leq k \leq N} v_{jk}, \quad (21)$$

where

$$\langle h_j \rangle = -\left(\frac{\hbar^2}{2M}\right) \nabla_j^2 + \bar{U}(\rho_j, z_j) \quad (22)$$

with the time averaged potential

$$\bar{U}(\rho, z) = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} U(\rho, \varphi - \Omega t, z) dt. \quad (23)$$

We note in passing that the time averaged TOP potential $\bar{U}(\rho, z)$ is not very well approximated by a simple anisotropic harmonic oscillator potential. More to our central point, there is no justification for employing the time averaged Hamiltonian $\langle H \rangle$ for atoms in a TOP trap. Eqs.(17), (18) and (20) provide (in a mathematically rigorous fashion) the appropriate Hamiltonian \mathcal{H} for a TOP trap, just as the general Eq.(5) has long provided the appropriate Hamiltonian for superfluid He^4 in a rotating bucket.[11]

In the Gross-Piteavskii[12,13] dilute quantum gas model of a Bose condensate, the TOP trap induced order parameter of the condensate ought to obey the an equilibrium equation which follows from our above considerations; It is

$$\left\{ -\frac{\hbar^2}{2M} \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) + i\hbar\Omega \frac{\partial}{\partial \varphi} + \hbar\gamma G \sqrt{\rho^2 + b^2 + 2b\rho \cos \varphi + 4z^2} \right\} \psi(\rho, \varphi, z) + \left(\frac{4\pi\hbar^2 a |\psi(\rho, \varphi, z)|^2}{M} \right) \psi(\rho, \varphi, z) = \mu \psi(\rho, \varphi, z), \quad (24)$$

where μ is the chemical potential, and a is the two body scattering length. The ground state order parameter Eq.(24) is more than a little complicated. However, some of the implications of Eq.(24) may be stated with confidence.

Only in the limit $\Omega \rightarrow 0$ can the Bose condensate order parameter $\psi(\mathbf{r})$ be chosen to be real. For the TOP trap situation with $\Omega \neq 0$, the order parameter is complex

$$\psi(\mathbf{r}) = \sqrt{n_s(\mathbf{r})} \exp(iM\Phi(\mathbf{r})/\hbar) \quad (25)$$

yielding via Eqs.(24) and (25) a non-trivial superfluid flow velocity $\mathbf{v}_s = \nabla\Phi$ and a non-zero mean angular momentum. The velocity flow is determined by a rotational quantum Bernoulli equation

$$\mu = M \left(\frac{1}{2} |\nabla\Phi|^2 - (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\Phi \right) + \tilde{U}, \quad (26)$$

where

$$\tilde{U} = U - \frac{\hbar^2}{2M} \left(\frac{\nabla^2 \sqrt{n_s}}{\sqrt{n_s}} \right) + \left(\frac{4\pi\hbar^2 a n_s}{M} \right). \quad (27)$$

If the TOP trap were not rotating, the small scattering length of two ^{87}Rb atoms ($a \sim 50\text{\AA}$), along with the small density of the condensate[4] n_s , would lead to a localization length within the trap potential[14] minimum determined by $d \sim (\hbar/M\gamma G)^{1/3} \sim .15\mu m$. This localization length d is much smaller than the magnetic length of $b \sim 8 \times 10^2 \mu m$, i.e. $d \ll b$. If the condensate stayed within a distance d of the minimum of the potential $U(\mathbf{r})$, then there would be frequent spin flips resulting in the loss of atoms from the trap. The rotating potential causes the condensate atoms climb a bit up the potential wall away from the center, acting in a manner closely analogous to particles in a centrifuge. Thus lifted from the potential minimum, the spin induced lifetimes of atoms in the trap are increased.

To see how the mechanics of the rotational equilibrium for the condensate density n_s may work, let us first make an analogy with a roulette wheel and a rolling metal ball found in many gambling casinos (not yet having completely entered the digital computer game simulation age). The mechanical roulette wheel turns and the ball is rolled into a grove of radius b . The ball rolls around this grove while the wheel spins. Along the grove, there are many minor potential minima, i.e. one minor minimum for every possible number that can be bet to win on the roulette wheel. The wheel continues to turn, and finally the ball falls into one of the potential minima. The ball then rotates with the same angular velocity as the roulette wheel achieving “rotational thermal equilibrium” at the winning number. The point is that the rotational velocity of the ball relative to the rotational velocity of the roulette wheel cannot differ from zero for a very long time. Otherwise, the bets would never get settled. The ball does *not* in the final stages of the game feel the “**T**ime average **O**f the **P**otential”, i.e that time average in which the ball is continually bouncing over the minor potential barriers during the initial stages of the game. The ball does *finally settle down to a smooth rotational free*

energy minimum.

In the TOP trap, the condensate is placed in a rotating potential groove of radius b . The Bose condensate can hardly stand still for very long while feeling only the “Time average Of the Potential”; i.e. the condensate can only shake rapidly up and down the z -axis for a limited amount of time. Then the condensate then starts to flow smoothly around the groove. The critical angular velocity for forming one quantum of circulation,

$$\kappa_0 = (2\pi\hbar/M) \sim 4.6 \times 10^{-5} \text{ cm}^2/\text{sec}$$

around to rotation axis cannot be very much larger than $\Omega_c \sim (\kappa_0/b^2)$ which is (in order of magnitude) how slowly the minute hand of a clock rotates. This is a much lower rotational velocity then the TOP angular velocity[13] of $(\Omega/2\pi) \sim 7.5 \times 10^3 \text{ Hz}$. One thus expects perhaps $\sim 10^5$ circulation quanta to flow around the axis of rotation. This circulation represents a very large number of quanta indeed. Unlike the roulette wheel there is but one potential minima per turn around the groove. The circulating Bose condensate, then flows up and down the potential as it forms in its own little oval flow (as in a toroidal pipe) around this funnel potential.

What happens to this flow of condensate around a toroidal pipe (so to speak) when the TOP trap is removed? The diameter of the pipe will increase and the tangential velocities to the flow around the ring will decrease (due to angular momentum conservation). The oval which is left, and displayed in an experimental picture could hardly be circular. It would be elliptic due to the displaced asymmetric axis of rotation. The above considerations are consistent with TOP trap experiments.

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